Linear Algebra II

11/04/2012, Wednesday, 9:00-12:00

1 (8+7=15 pt) **Inner product spaces**

(a) Consider the vector space $\mathbb{R}^n$. Show that $\langle x, y\rangle = x^T M y$ is an inner product if and only if $M \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix.

(b) Let V be an inner product space and let $\|v\| = \sqrt{\langle v, v\rangle}$. Prove that

$$\|x - y\|^2 + \|x + y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

for all $x, y \in V$.

2 (2+2+5+6=15 pt) **Orthogonal matrices**

Let $A \in \mathbb{R}^{n\times n}$.

(a) Show that if $(I + A)$ is nonsingular then $(I - A)(I + A)^{-1} = (I + A)^{-1}(I - A)$.

(b) Show that if $A = -A^T$ then $(I + A)$ is nonsingular.

(c) Show that if $A = -A^T$ then $(I - A)(I + A)^{-1}$ is an orthogonal matrix.

(d) Show that if A is orthogonal and $(I + A)$ is nonsingular then $B = -B^T$ where $B = (I - A)(I + A)^{-1}$

3 (8+7=15 pt) **Diagonalization and positive definite matrices**

Let

$$A = \begin{bmatrix} a & b & 0 \\ c & d & c \\ 0 & b & a \end{bmatrix}$$

where a, b, c, and d are real numbers.

(a) For which values of (a, b, c, d) is the matrix A unitarily diagonalizable?

(b) For which values of (a, b, c, d) is the matrix A positive definite? (Warning: The matrix A is not necessarily symmetric!)

4 (8+7=15 pt) — Cayley-Hamilton theorem

(a) Let $v \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$. Show that the subspace $\operatorname{span}\{v, Av, \ldots, A^{n-1}v\}$ is invariant under A.

(b) Let

$$M = \begin{bmatrix} 0 & 0 & -1+a \\ 1 & 0 & -3 \\ 0 & 1 & -3 \end{bmatrix}.$$

Find $(M+I)^{3000}$.

5 (15 pt) — Singular value decomposition

Find a singular value decomposition for the matrix

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

and its best rank 1 approximation in the Frobenius norm.

6 (15 pt) — Jordan canonical form

Put the matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}.$$

into Jordan canonical form.

Hint: Note that $(a \pm 1)^3 = a^3 \pm 3a^2 + 3a \pm 1$.

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10 pt gratis